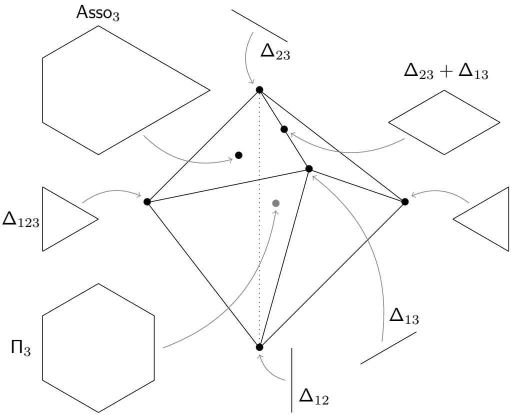
Deformed permutahedra
Let's visit the submodular cone together

Germain Poullot



27 Febuary 2025

1. Deformations (a.k.a. Minkowski summands)
   - Minkowski decomposition
   - Cone of deformations
2. Generalized permutahedra as deformations
   - Combinatorics of the permutahedron
   - Submodular functions
3. Submodular Cone in general
   - Known facts about $\mathbb{SC}_n$
   - Ongoing work
4. Submodular cone $n = 4$ (and $n = 5$)
   - Drawing and quotienting $\mathbb{SC}_4$
   - About rays of $\mathbb{SC}_n$
   - Fun facts!

Deformations (a.k.a. Minkowski summands)

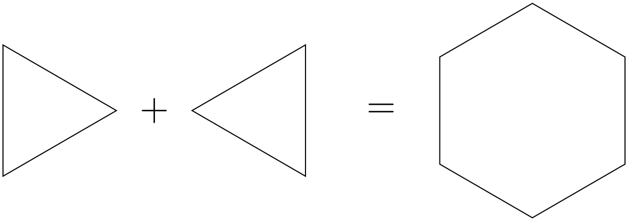
Minkowski sum

Definition

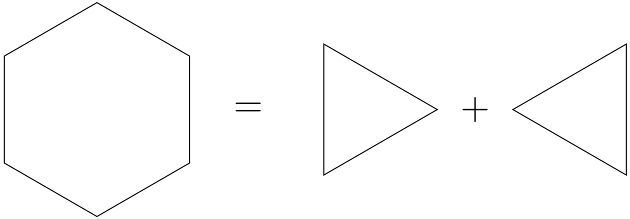
P, Q polytopes. *Minkowski sum*:

$$\mathrm{P} + \mathrm{Q} = \{\boldsymbol{p} + \boldsymbol{q} \;\; ; \;\; \boldsymbol{p} \in \mathrm{P}, \;\; \boldsymbol{q} \in \mathrm{Q}\}$$

N.B. $\mathrm{Vert}(\mathrm{P} + \mathrm{Q}) \subseteq \mathrm{Vert}(\mathrm{P}) + \mathrm{Vert}(\mathrm{Q})$



Minkowski sum



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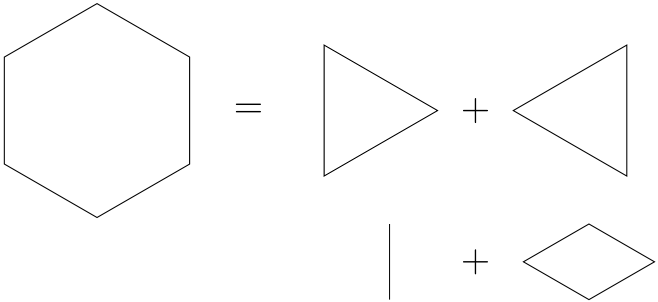

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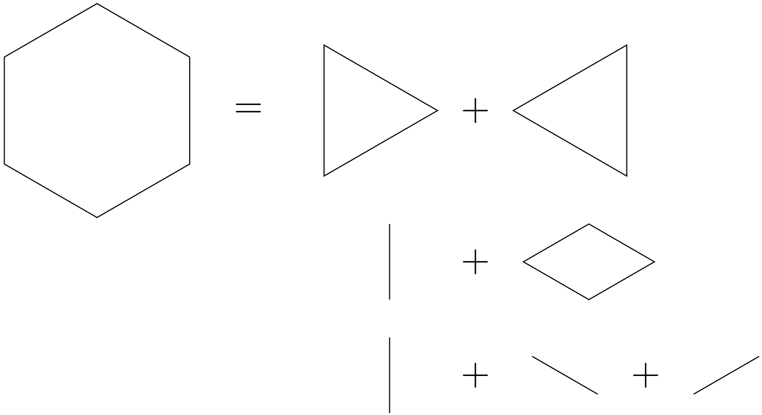

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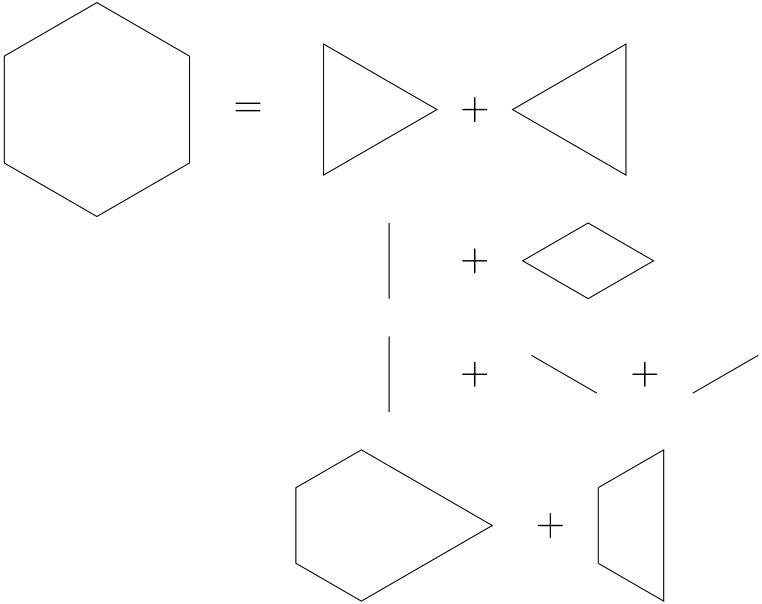

Minkowski summands

Definition

Q is a *Minkowski summand*, a.k.a. *deformation*, of P when there exists R and $\lambda > 0$ with:

$$\lambda P = Q + R$$

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Minkowski indecomposable: deformations of P are dilations of P

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What is the best way to write P as a Minkowski sum ?

- With the fewest number of (indecomposable) summands ?
- With the (indecomposable) summands of smallest dimension ?
- Respecting some symmetries ?
- ...

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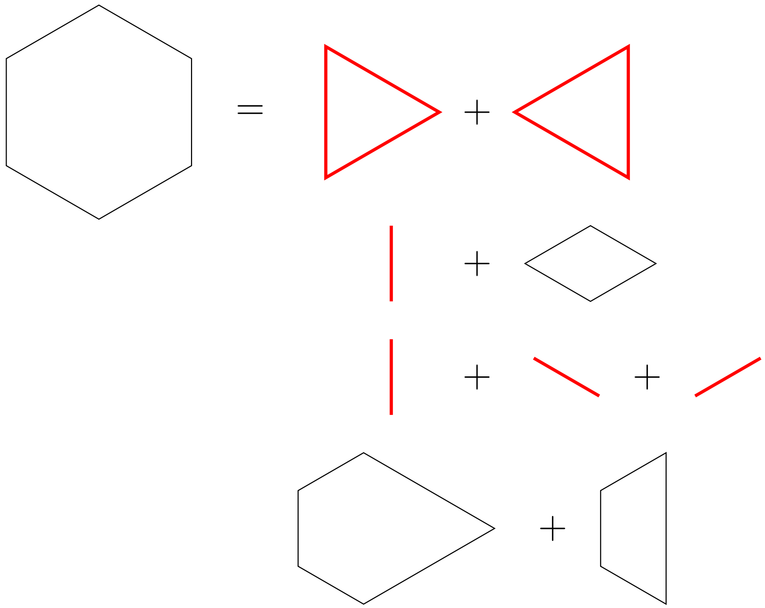
$\Longrightarrow$ What is the structure of $\mathbb{DC}(\mathsf{P})$?

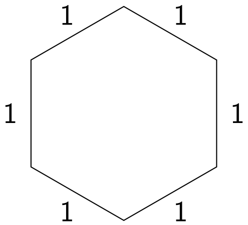
Minkowski summands

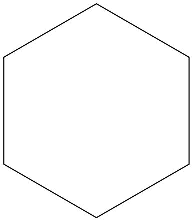


Deformation

Observation

If $P = Q + R$, then the edges of P "are" edges of Q or of R.
$\Rightarrow$ I can write deformations of P using edge-length vectors.





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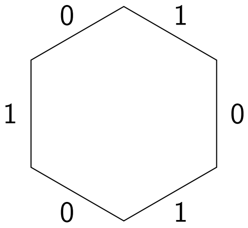

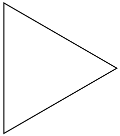

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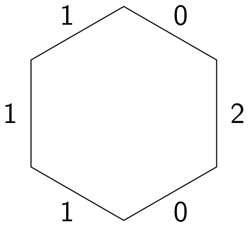

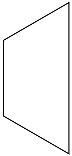

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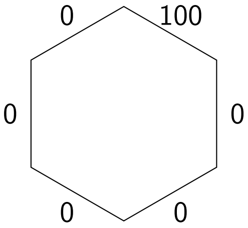
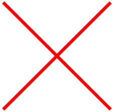
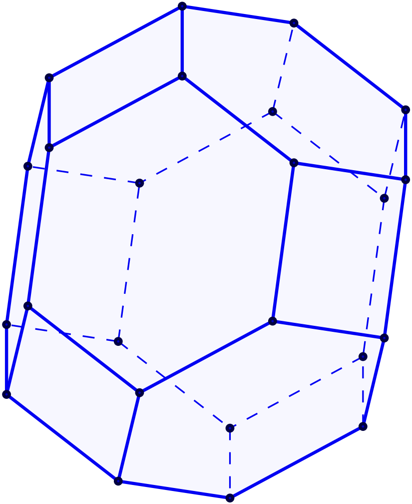
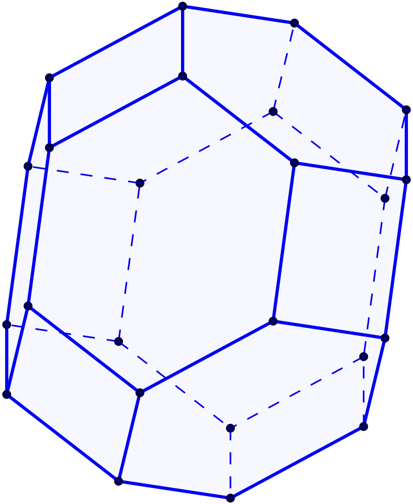




Deformations of 3-dim permutahedron



Permutahedron Π_4



Sequence of deformations of Π_4

Edge-length deformation cone

Theorem

Q *deformation of* P $\Leftrightarrow$ *same edge-directions, but different lengths*

Definition

Edge-length deformation cone: $\mathbb{DC}(\mathsf{P}) = \{\mathsf{Q}\ ;\ \mathsf{Q} \text{ same edge-dir } \mathsf{P}\}$

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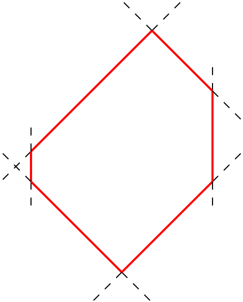

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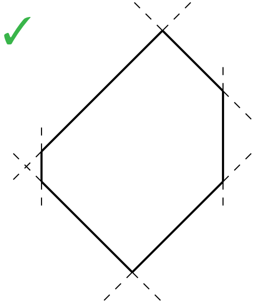

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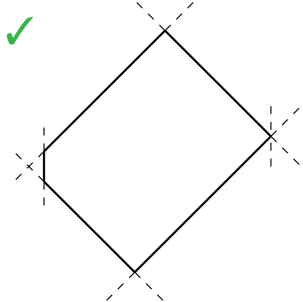

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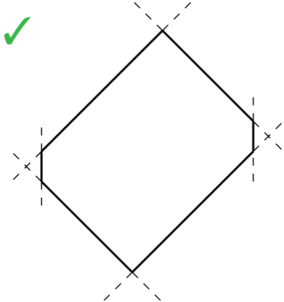

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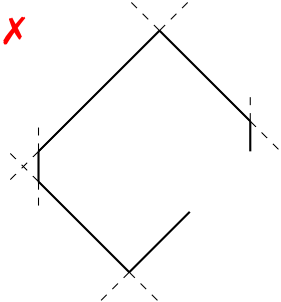

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Polygonal face equations:

linear equations on $\boldsymbol{\ell}$

$\ell_{\boldsymbol{e}} \geq 0$ for all $\boldsymbol{e}$ edge

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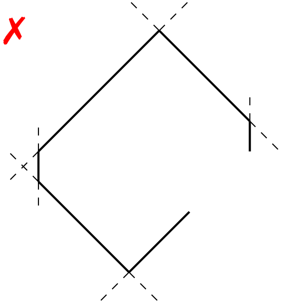

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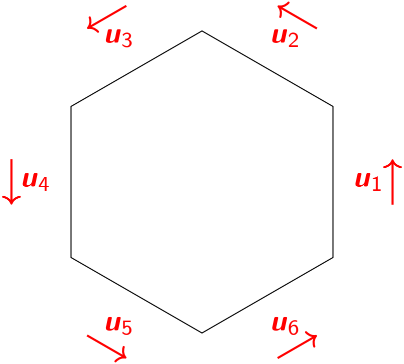
$\mathrm{P}_{\boldsymbol{\ell}}$ = start at a vertex, find the coordinates of the other vertices from the graph of P and $\boldsymbol{\ell}$

Polygonal face equations



For F a 2-dim face of P:

$$\sum_{\boldsymbol{e}} \boldsymbol{u}_{\boldsymbol{e}} = \mathbf{0} \quad , \quad \boldsymbol{u}_{\boldsymbol{e}} \text{ unit vector}$$

hence

$$\sum_{\boldsymbol{e}} \ell_{\boldsymbol{e}} \boldsymbol{u}_{\boldsymbol{e}} = \mathbf{0}$$

Summary on $\mathbb{DC}(P)$

$\mathbb{DC}(P)$

Q	$\boldsymbol{\ell}$	$\boldsymbol{h}$
Minkowski summands	edge-lengths	heights on rays
$Q_1 + Q_2$	$\boldsymbol{\ell}_1 + \boldsymbol{\ell}_2$	$\boldsymbol{h}_1 + \boldsymbol{h}_2$
Dilation λQ	$\lambda\boldsymbol{\ell}$	$\lambda\boldsymbol{h}$
Translations	Has been fixed	Lineal
	edge directions	normal fan $\mathcal{N}_P$
complicated	Polygonal face eq.	Wall-crossing ineq.
	V-description	H-description
Polytope algebra	Weight algebra	Polynomial algebra

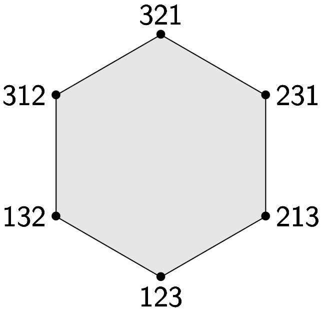
$\mathbb{DC}(P)$ is a ray = P Minkowski indecomposable

$\mathbb{DC}(P)$ is simplicial cone = P has **unique** Minkowski decomposition

Generalized permutahedra as deformations

Example (Permutahedron)

$$\Pi_n = \text{conv}\left\{ \begin{pmatrix} \sigma(1) \\ \vdots \\ \sigma(n) \end{pmatrix} \; ; \; \sigma \text{ permutation of } \{1, \dots, n\} \right\}$$



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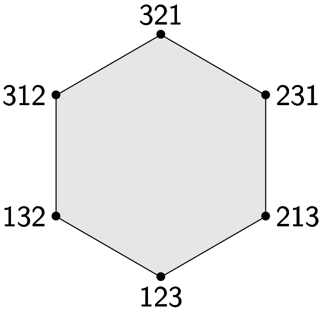
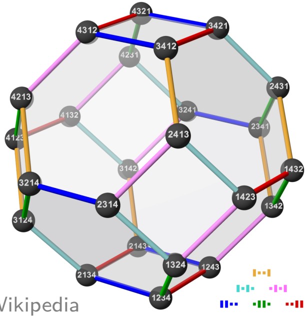




Wikipedia

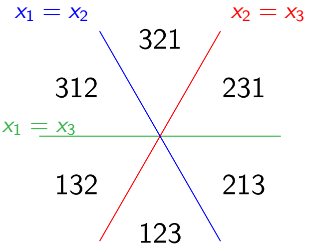
Braid fan

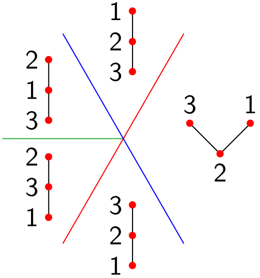
Definition

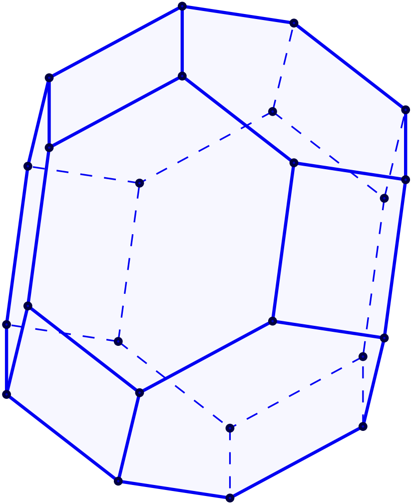
Generalized permutahedron: deformation of Π_n

i.e. P generalized permutatahedron iff edges in directions $\boldsymbol{e}_i - \boldsymbol{e}_j$

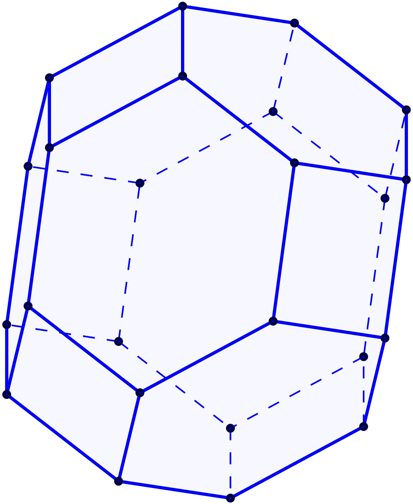
i.e. P generalized permutahedron iff $\mathcal{N}_{\mathrm{P}}$ coarsens braid fan



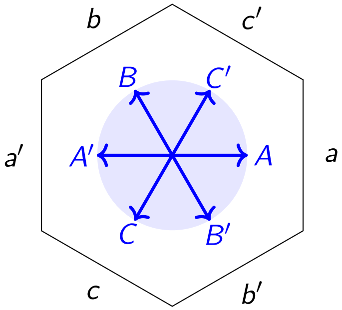




Permutahedron Π_4



Sequence of deformations of Π_4



Wall-crossing inequalities:

$h_A + h_B \geq h_{C'}$

$h_B + h_C \geq h_{A'}$

$h_C + h_A \geq h_{B'}$

& 3 others ineq.

Polygonal face equations:

$\ell_a - \ell_{a'} = \ell_b - \ell_{b'} = \ell_c - \ell_{c'}$

& $\boldsymbol{\ell} \in \mathbb{R}^6_+$

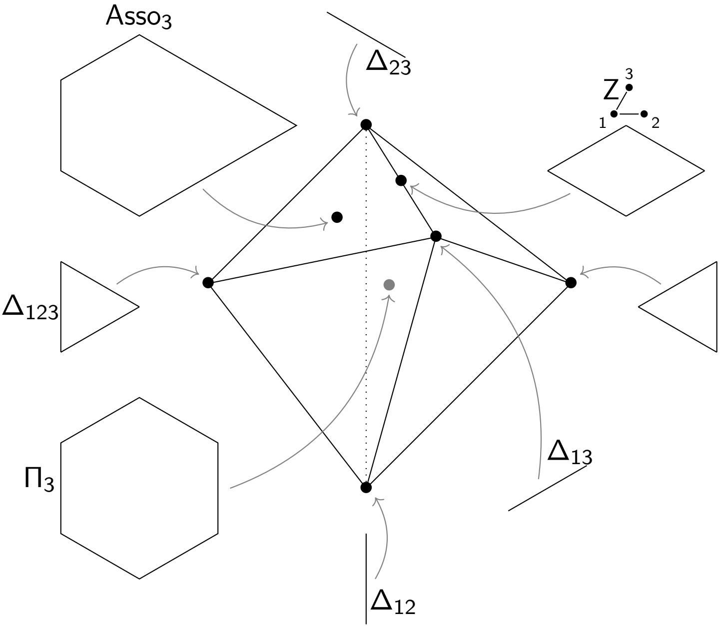
Submodular Cone in general

Definition

Submodular cone: deformation cone of the permutahedron Π_n

	$\mathbb{DC}(\Pi_n)$
Dim (no lineal)	$2^n - n - 1$
# facets	$\binom{n}{2}2^{n-2}$
# rays	unknown!

Submodular Cone for Π_3



Submodular Cone's faces

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Submodular cone $\mathbb{SC}_n$: deformation cone of the permutahedron Π_n

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Theorem (Faces of $\mathbb{DC}(\mathsf{P})$)

If Q *deformation of* P*, then* $\mathbb{DC}(\mathsf{Q})$ *is a face of* $\mathbb{DC}(\mathsf{P})$*.*

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	$\mathbb{DC}(\Pi_n)$	$\mathbb{DC}(\mathrm{Asso}_n)$
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# facets	$\binom{n}{2}2^{n-2}$	$\binom{n}{2}$
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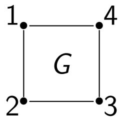
	$\mathbb{DC}(\Pi_n)$	$\mathbb{DC}(\mathrm{Asso}_n)$	$\mathbb{DC}(\mathrm{Z}_G)$	$\mathbb{DC}(\mathrm{N}_B)$
Dim (no lineal)	$2^n - n - 1$	$\binom{n}{2}$	N	N
# facets	$\binom{n}{2}2^{n-2}$	$\binom{n}{2}$	E	E
# rays	unknown!	$\binom{n}{2}$	X	X
		is simplicial!	T	T

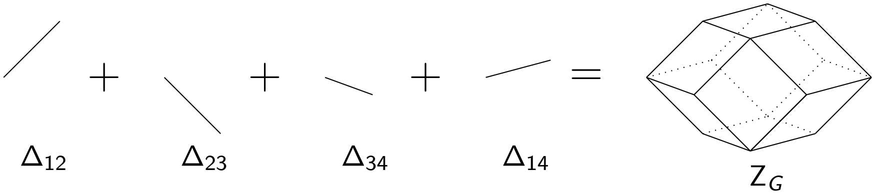
$G = (V, E)$ a graph, $n = |V|$

Definition

Graphical zonotope $\mathrm{Z}_G := \sum_{(i,j)\in E} [\boldsymbol{e}_i, \boldsymbol{e}_j]$

Z_G deformation of $\Pi_n \implies \mathbb{DC}(\mathrm{Z}_G)$ is a face of $\mathbb{DC}(\Pi_n)$





Theorem (Padrol, Pilaud, P., '23)

Explicit facet-description of $\mathbb{DC}(\mathsf{Z}_G)$

Graphical Zonotopes

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Explicit facet-description of $\mathbb{DC}(\mathsf{Z}_G)$

Corollary

dim $\mathbb{DC}(\mathsf{Z}_G) = \#$ *cliques of* G

$\#$ *facets of* $\mathbb{DC}(\mathsf{Z}_G) = \sum_{(i,j)\in E} 2^{|\{k \;;\; (i,k),(j,k)\in E\}|}$

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$\mathbb{DC}(\mathsf{Z}_G)$ *simplicial iff* G *without triangle*

NB: Recover facet-description of $\mathbb{DC}(\Pi_n)$

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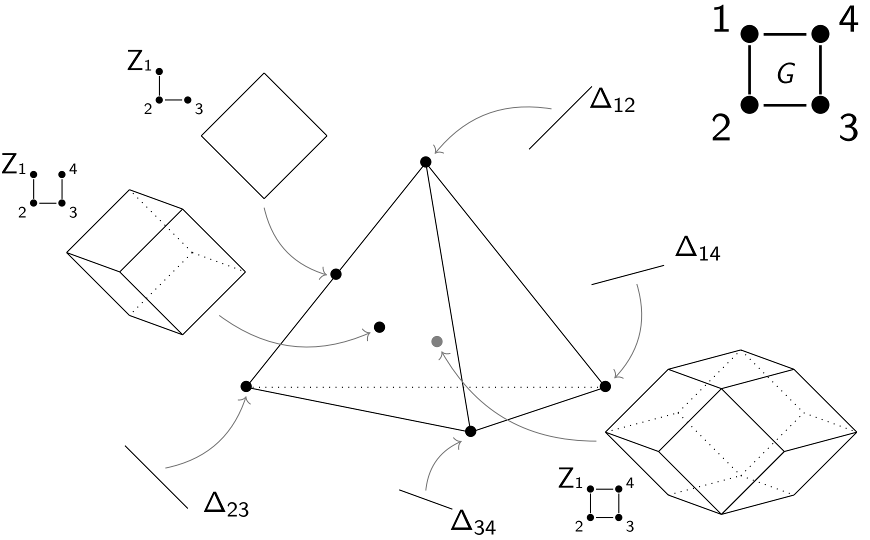
$\mathbb{DC}(\mathsf{Z}_G)$ *simplicial iff G without triangle*

NB: Recover facet-description of $\mathbb{DC}(\Pi_n)$

Theorem (P., '24)

If G is K_4*-free, then all rays of* $\mathbb{DC}(\mathsf{Z}_G)$ *are 1- and 2-dimensional.*

Graphical Zonotopes



Definition

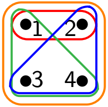
Building set $B \subseteq 2^{[n]}$ with: $X_{1,2} \in B, X_1 \cap X_2 \neq \emptyset \Rightarrow X_1 \cup X_2 \in B$

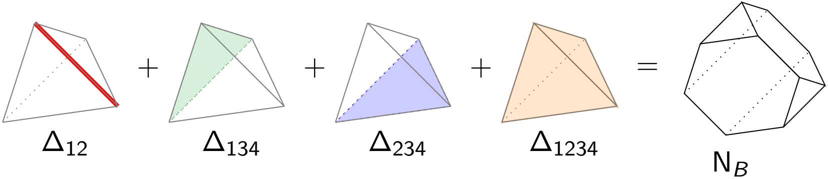
Definition

Nestohedron $\mathrm{N}_B := \sum_{X \in B} \Delta_X$ where $\Delta_X = \mathrm{conv}\{\boldsymbol{e}_i \ ; \ i \in X\}$

N_B deformation of $\Pi_n \implies \mathbb{DC}(\mathrm{N}_B)$ is a face of $\mathbb{DC}(\Pi_n)$

B





Nestohedra

Elementary blocks $X \in \varepsilon(B)$ iff X is not a union

Maximal block $\mu(X) := \max\{Y \in B\ ;\ Y \subsetneq X\}$

Theorem (Padrol, Pilaud, P., '23)

Explicit facet description of $\mathbb{DC}(\mathsf{N}_B)$

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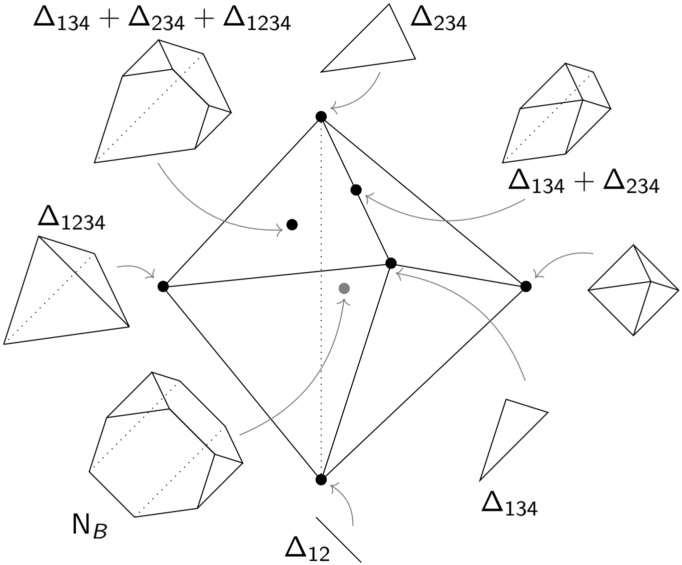
Corollary

$\mathbb{DC}(\mathsf{N}_B)$ *simplicial iff* B *has no non-elementary block with 3 maximal subblocks*

NB: Recover facet-description of $\mathbb{DC}(\Pi_n)$

Nestohedra

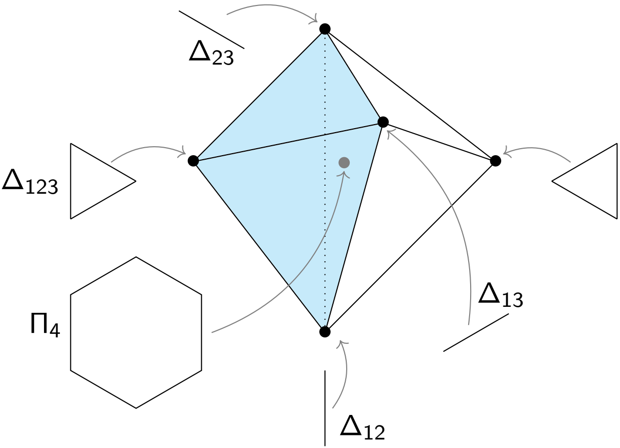


$\Delta_{134} + \Delta_{234} + \Delta_{1234}$
Δ_{234}
$\Delta_{134} + \Delta_{234}$
Δ_{1234}
N_B
Δ_{12}
Δ_{134}

Ongoing work - Hypergraphic polytopes

Definition

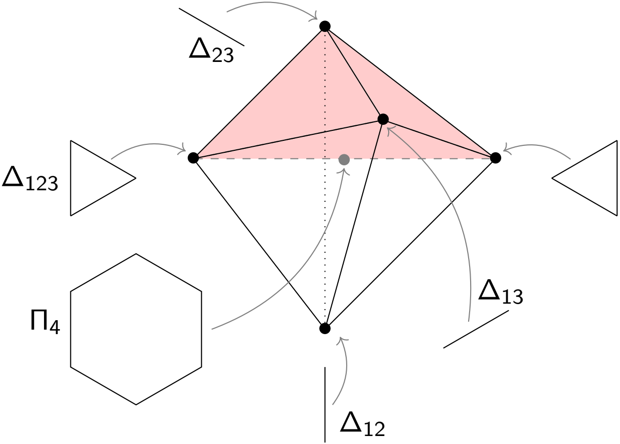
Hypergraphic polytope $\mathrm{P}_H := \sum_{X \in H} \Delta_X \quad$ with $H \subseteq 2^{[n]}$



Ongoing work - Quotientopes

Definition

Quotientopes: Minkowski sum of shard polytopes



Submodular cone $n = 4$ *(and* $n = 5$*)*

Drawing $\mathbb{SC}_4$

Recall: dim $= 11$, nbr facets $= 80$

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Draw all generalized permutahedra ? (ask computer)

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22 107 faces (Please do not draw...)

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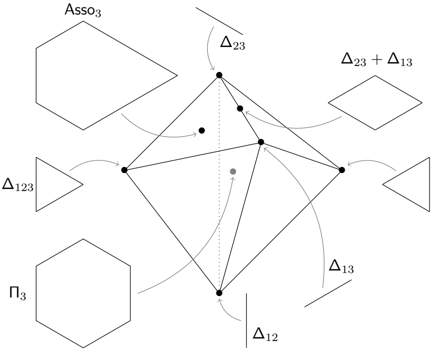
Draw all generalized permutahedra ? (ask computer)

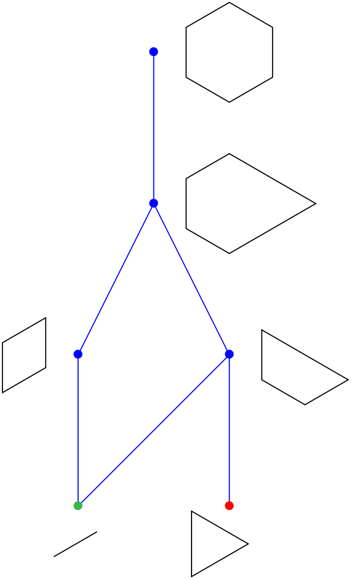
22 107 faces (Please do not draw...)

$\Longrightarrow$ quotient by symmetries

Symmetries of the braid fan

Braid symmetries: permutation of coordinates + central symmetry





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Drawing $\mathbb{SC}_4$

Recall: dim $= 11$, nbr facets $= 80$

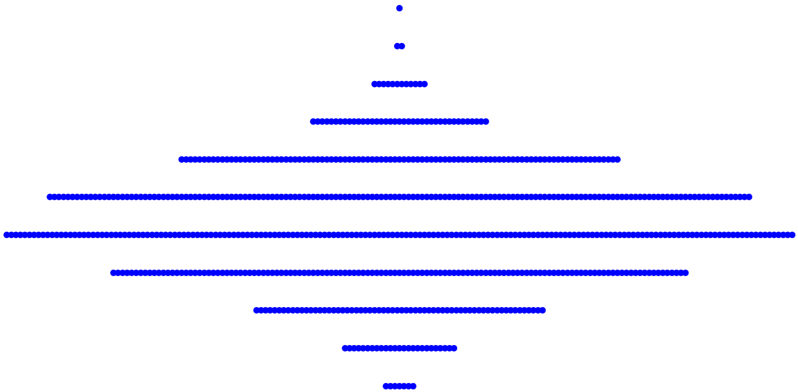
Draw all generalized permutahedra ? (ask computer)

22 107 faces

$\Longrightarrow$ quotient by symmetries

703 "faces"

Reduced face lattice of $\mathbb{SC}_4$



d	1	2	3	4	5	6	7	8	9	10	11	total
f_d	7	25	64	127	174	155	97	39	12	2	1	703

Reduced f-vector of $\mathbb{SC}_n$

Reduced $\mathbb{SC}_n$ f-vector:

$n = 3$
dim $\mathbb{SC}_3 = 4$
$(2, 2, 1, 1)$

$n = 4$, dim $\mathbb{SC}_4 = 11$
$(7, 25, 64, 127, 174,$
$155, 97, 39, 12, 2, 1)$

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dim $\mathbb{SC}_3 = 4$
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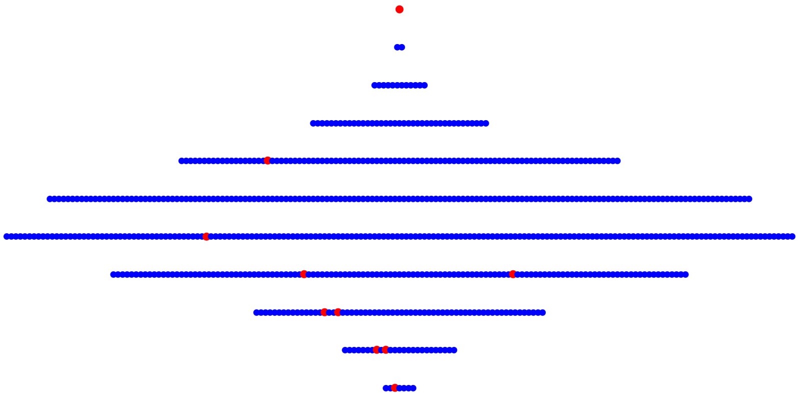
$(7, 25, 64, 127, 174,$
$155, 97, 39, 12, 2, 1)$

Thanks to Winfried Bruns for helping with computations!

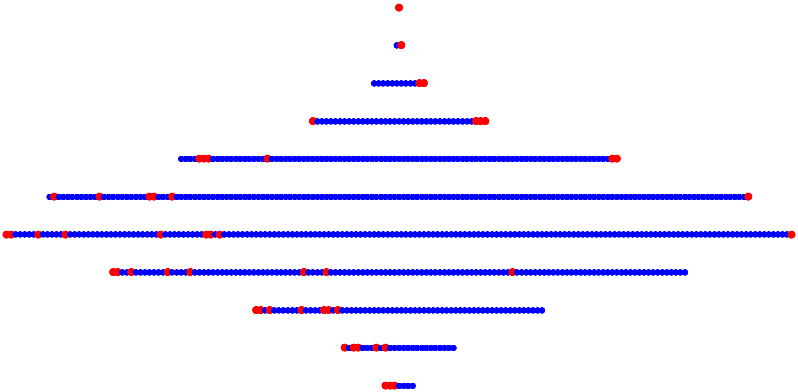
Database for dim 1-4 & 19-26

$n = 5$, dim $\mathbb{SC}_5 = 26$
∗672
∗24 026
∗373 433
∗3 355 348
19 739 627
81 728 494
249 483 675
579 755 845
1 048 953 035
1 501 555 944
1 719 688 853
1 587 510 812
1 186 372 740
719 012 097
353 190 577
140 265 886
44 831 594
11 464 559
∗2 326 596
∗372 031
∗46 330
∗4 572
∗355
∗30
∗2
∗1



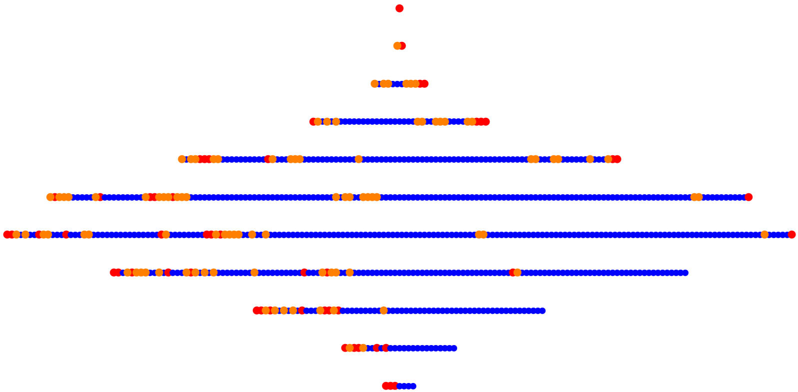
With: Graphical Zonotopes
10 polytopes

Graphical zonotopes & Nestohedra are sparse



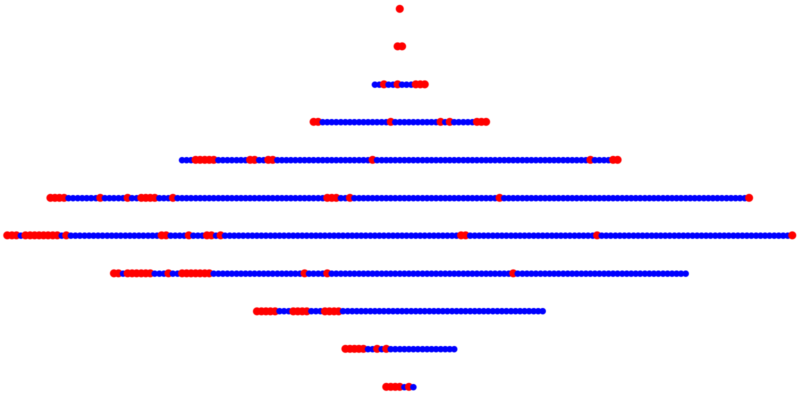
With: Graphical Zonotopes & Nestohedra
10 + 46 polytopes

Graphical zonotopes & Nestohedra are sparse



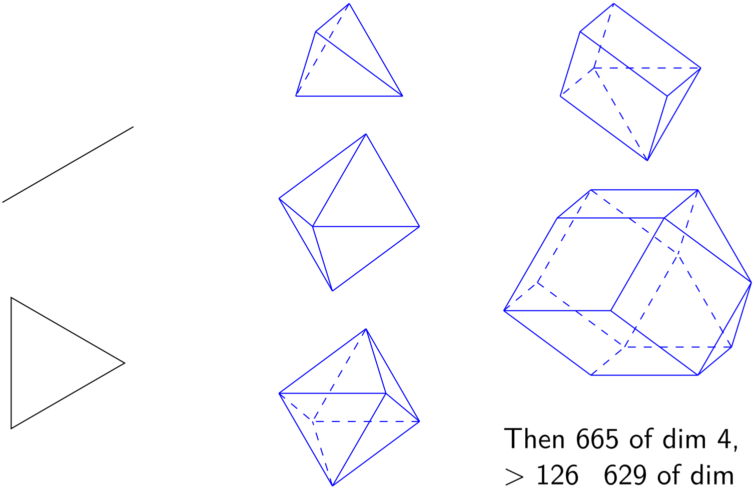
With: Graphical Zonotopes & Nestohedra + facets
147 polytopes in total

Everything is quite negligible...



With: Graphical Zono & Nestohedra $\subsetneq$ Hypergraphic Polytopes,
$+$ Shard Polytopes, Quotientopes,
$+$ Matroid Polytopes
$=$ 112 polytope (only...)

What about the rays of $\mathbb{SC}_4$?



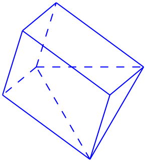
Then 665 of dim 4, $> 126\ 629$ of dim 6…

Strawberry & Persimmon

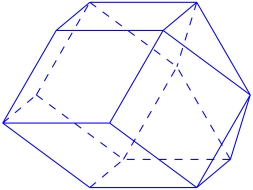
Example of GP:

Minkowski indecomposable ✔

Matroid Polytopes ✗



Strawberry



Persimmon

Strawberry & Persimmon

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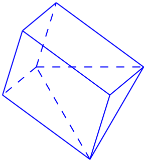
Matroid Polytopes ✗
Hypergraphic polytopes ✗
Shard Polytopes ✗



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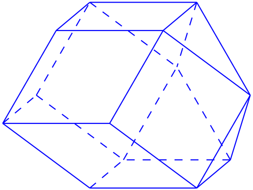


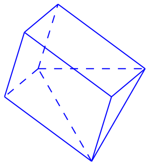
Persimmon

Example of GP:

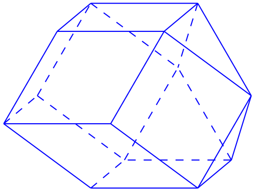
Minkowski indecomposable ✔

Matroid Polytopes ✗
Hypergraphic polytopes ✗
Shard Polytopes ✗

Persimmon:
Polypositroid ✗
Removahedron ✗



Strawberry



Persimmon

Ray of $\mathbb{SC}_n$ = indecomposable generalized permutahedron

Theorem (Nguyen, '78)

The rays of $\mathbb{SC}_n$ *which uses only* 0/1*-coordinates are known. There are:*

$$\#\text{rays of } \mathbb{SC}_n \geq 2^{2^{n-\frac{3}{2}\log n+O(1)}}$$

How many rays of $\mathbb{SC}_n$?

Ray of $\mathbb{SC}_n$ = indecomposable generalized permutahedron

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The rays of $\mathbb{SC}_n$ which uses only 0/1-coordinates are known. There are:

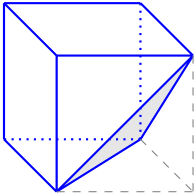
$$\#\textit{rays of } \mathbb{SC}_n \geq 2^{2^{n-\frac{3}{2}\log n + O(1)}}$$

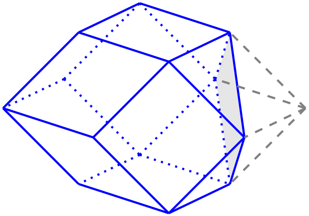
Theorem (consequence of Rosenmüller, Weidner, '73)

There exists some rays which are not on 0/1-coordinates.

Theorem (Padrol, P., '25$^+$ $\to$ come to FPSAC 2025!)

Truncate vertices of $Z_{K_{k,m}}$, *you can get* $2\left\lfloor \frac{n-1}{2} \right\rfloor$ *new rays of* $\mathbb{SC}_n$.





Fun fact 1: Smilanski's conjecture

Smilanski's conjecture, '87

If P is indecomposable with $\dim \mathrm{P} = 4$, then $f_0 < 2f_{d-1} - 4$.

FALSE!

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Smilanski's conjecture, '87

If P is indecomposable with $\dim \mathrm{P} = 4$, then $f_0 < 2f_{d-1} - 4$.

FALSE!

Observation

There are 84 counter-examples in the database.
There exists an indecomposable GP with f-vector:

$$(\mathbf{66}, 153, 113, \mathbf{26})$$

Fun fact 2: Edge lengths

Equilateral: all edges have same length

Observation

All Minkowski indecomposable GP in $\mathbb{SC}_4$ are equilateral

Fun fact 2: Edge lengths

Equilateral: all edges have same length

Observation

All Minkowski indecomposable GP in $\mathbb{SC}_4$ are equilateral

Observation

Exists Minkowski indecomposable GP in $\mathbb{SC}_5$ **not** equilateral

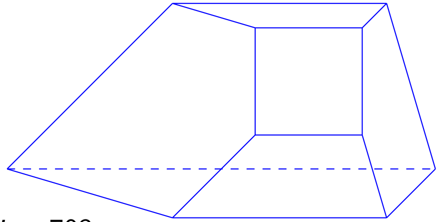
For $n = 5$

# edge-length classes	1	2	3	4	5	6
# Minkowski indec GP	41	292	250	73	12	4

Usefull for proving Minkowski indecomposabilitiy

Fun fact 3: Combinatorial equivalence

Two GP can be
combinatorially eq. ✔
eq. up to symmetries ✗



For $n = 4$
Up to symmetries: 703
Up to combinatorial eq.: 532

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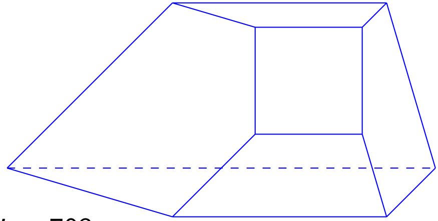


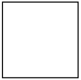
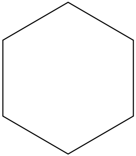
For $n = 4$
Up to symmetries: 703
Up to combinatorial eq.: 532

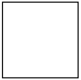
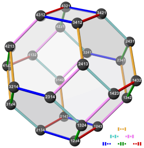
Conjecture

Minkowski indecomposable GP + combinatorially eq.
$\Rightarrow$ eq. up to symmetries

True up to $n = 5$ (i.e. 672 examples) and for 126 629 example of $n = 6$

Fun fact 4: Dimensions of GP

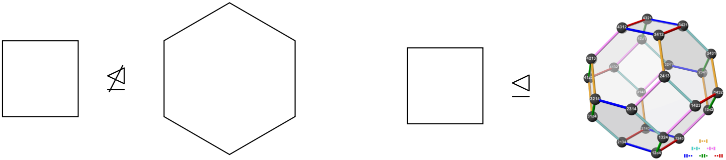
 $\ntrianglelefteq$ 

 $\trianglelefteq$ 

Observation

Exists some GP : $\dim P = n - 1$ but $P \ntrianglelefteq \Pi_n$

Fun fact 4: Dimensions of GP



Observation

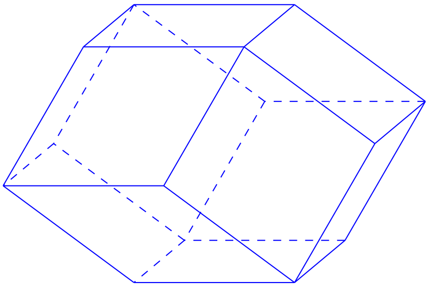
Exists some GP : $\dim P = n - 1$ but $P \ntrianglelefteq \Pi_n$

Conjecture

P is Minkowski indecomposable + $\dim P = n - 1$
$\Rightarrow \quad P \trianglelefteq \Pi_n$

True up to $n = 5$ (i.e. 672 examples) and for 126 629 example of $n = 6$

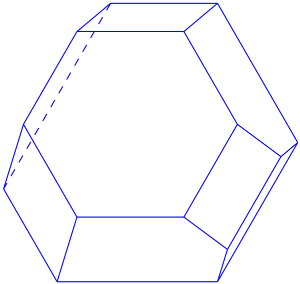
Fun fact 5: Non hamiltonian GP

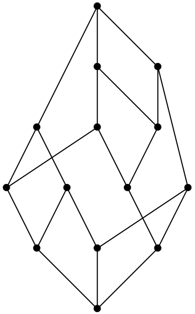


Observation

Exists GP graph with no hamiltonian path (1 in $\mathbb{SC}_4$)
Exists GP graph with no hamiltonian cycle (9 in $\mathbb{SC}_4$)

Fun fact 6: GP with lattice graph



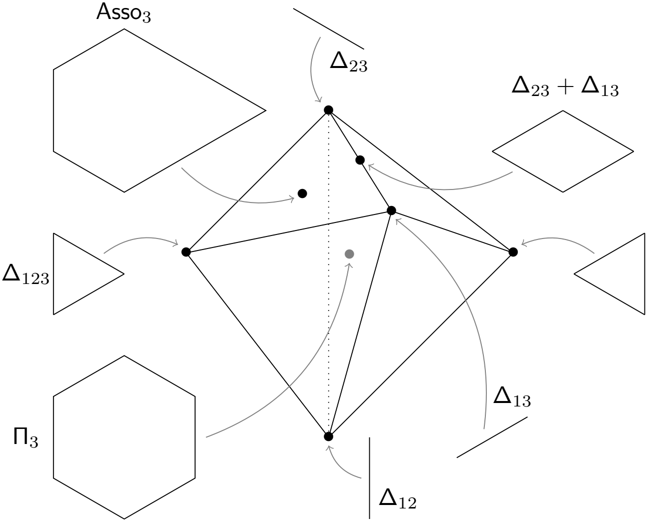


Observation

Exists GP not quotientopes[a] but its oriented graph is a lattice
(339 in $\mathbb{SC}_4$, 27 are simple)

[a] i.e. not combinatorially eq. to a quotientope

Thank you!



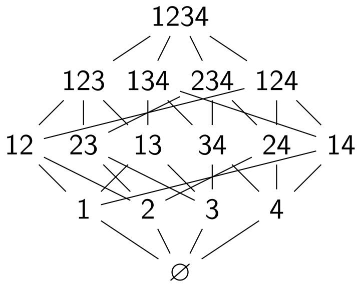
Asso_3
Δ_{23}
$\Delta_{23} + \Delta_{13}$
Δ_{123}
Δ_{13}
Π_3
Δ_{12}

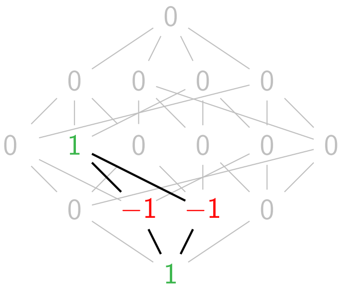
Tools: submodular dependancies

Notations: $Sx = S \cup \{x\}$, $(\boldsymbol{f}_X)_{X \subseteq [n]}$ canonical basis of $\mathbb{R}^{2^{[n]}}$

Definition

Submodular vector $\boldsymbol{n}(S, u, v) = \boldsymbol{f}_{Suv} - \boldsymbol{f}_{Su} - \boldsymbol{f}_{Sv} + \boldsymbol{f}_S$
for $u, v \in S \subseteq [n]$





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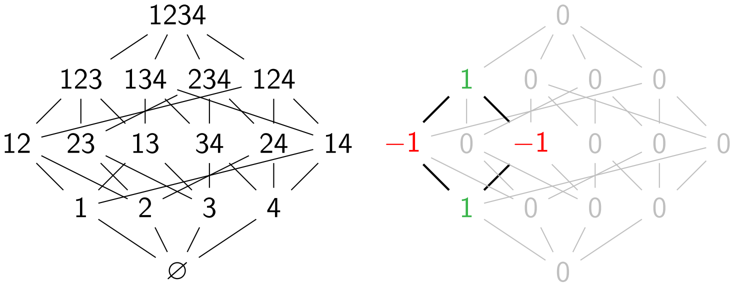

Lemma (Submodular normal)

$\boldsymbol{n}(S,u,v)$ *are the facet's normals of* $\mathbb{DC}(\Pi_n)$

Lemma (Cubic relation)

$u, v, x \notin S \subseteq [n]$

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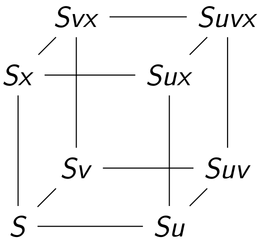

NB: Cubic relations generates all relations of submodular vectors

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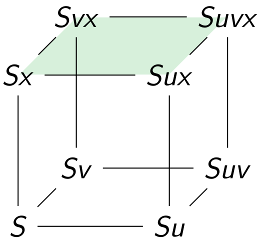

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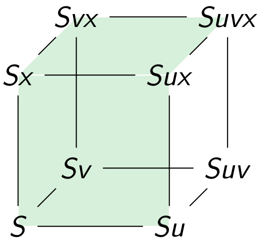

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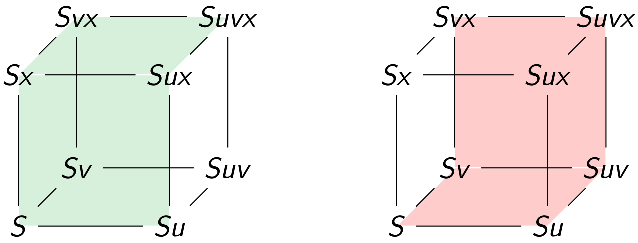

NB: Cubic relations generates all relations of submodular vectors

Tools: Shepard–McMullen indecomposability criterions

Theorem (Shepard, '63)

If every two vertices of P *are joined by a strong chain of indecomposable faces, then* P *is indecomposable.*

Theorem (McMullen, '87)

If a (strongly connected family of) indecomposable face(s) touches every facets, then P *is indecomposable.*

Improvement of McMullen's criterion:

Theorem (Loho, Padrol, P., '24+)

If a (strongly connected family of) indecomposable face(s) touches every facets except F*, and* F *has a vertex with two neighbors outside of* F*, then* P *is indecomposable.*

Multiple choice questions
(sometimes several answer possible)

A sum of 2 co-planar triangles can have	3 edges	4 edges	5 edges	6 edges
$\mathcal{N}_{\mathsf{P+Q}} =$ ___ of $\mathcal{N}_{\mathsf{P}}, \mathcal{N}_{\mathsf{Q}}$	union	intersection	common refinement	product
Edge directions of GP $\in$	$\{\boldsymbol{e}_i\}_i$	$\{\boldsymbol{e}_i + \boldsymbol{e}_j\}_{i,j}$	$\{\boldsymbol{e}_i - \boldsymbol{e}_j\}_{i,j}$	$\{\boldsymbol{e}_i \pm \boldsymbol{e}_j\}_{i,j}$
Z_G has no hexagonal face iff G is	a tree	K_3-free	K_4-free	complete
___ form a basis of $\mathrm{Vect}(\mathbb{SC}_n)$	$\{\Delta_X\}_{X\subseteq[n]}$	nestohedra	shard polytopes	matroid polytopes
$\dim \mathbb{SC}_3 =$	3	4	5	11
Group of symmetries of $\mathbb{SC}_n$:	Free_n	$\mathcal{S}_n$	$\mathcal{S}_n \times \mathbb{Z}_2$	$\mathcal{S}_n \times \mathbb{Z}_n$
# 3-dim indecomposable GP:	3	5	7	37

Exercises

1. Show that $\mathrm{Vert}(\mathsf{P}+\mathsf{Q}) \subseteq \{\boldsymbol{u}+\boldsymbol{v} \;;\; \boldsymbol{u} \in \mathrm{Vert}(\mathsf{P}),\; \boldsymbol{v} \in \mathrm{Vert}(\mathsf{Q})\}$. Find an example with strict inclusion. Prove that $\mathcal{N}_{\mathsf{P}+\mathsf{Q}}$ is the common refinement of $\mathcal{N}_{\mathsf{P}}$ and $\mathcal{N}_{\mathsf{Q}}$.

2. Find all the ways to write the 2-dimensional permutahedron as a Minkowski sum of indecomposable polytopes.

3. Prove that $\boldsymbol{h}_{\mathsf{P}+\mathsf{Q}} = \boldsymbol{h}_{\mathsf{P}} + \boldsymbol{h}_{\mathsf{Q}}$ and $\boldsymbol{\ell}_{\mathsf{P}+\mathsf{Q}} = \boldsymbol{\ell}_{\mathsf{P}} + \boldsymbol{\ell}_{\mathsf{Q}}$. What are $\boldsymbol{h}_{\mathsf{P}+\boldsymbol{t}}$ and $\boldsymbol{\ell}_{\mathsf{P}+\boldsymbol{t}}$?

4. Prove that a triangle is Minkowski indecomposable. Deduce that all simplicial polytopes are Minkowski indecompsable.

Exercises

5. Take vectors $\boldsymbol{s}, \boldsymbol{s}', \boldsymbol{r}_1, \dots, \boldsymbol{r}_{d-1}$ such that $\mathrm{C} = \mathrm{cone}(\boldsymbol{s}, \boldsymbol{r}_1, \dots, \boldsymbol{r}_{d-1})$ and $\mathrm{C}' = \mathrm{cone}(\boldsymbol{s}', \boldsymbol{r}_1, \dots, \boldsymbol{r}_{d-1})$ are simplicial cones which intersect on their proper face $\mathrm{C} \cap \mathrm{C}' = \mathrm{cone}(\boldsymbol{r}_1, \dots, \boldsymbol{r}_{d-1})$. Show that there exist $\alpha_{\boldsymbol{s}}, \alpha_{\boldsymbol{s}'} > 0$ and $\alpha_{\boldsymbol{r}_i} \in \mathbb{R}$ such that:

$$\alpha_{\boldsymbol{s}} \boldsymbol{s} + \alpha_{\boldsymbol{s}'} \boldsymbol{s}' + \sum_i \alpha_{\boldsymbol{r}_i} \boldsymbol{r}_i = \boldsymbol{0}$$

6. Compute (and draw) the deformation cone of a parallelogram (use the edge-lengths point of view). What about a parallelopiped? Show that there is a **unique** way to write a parallelopiped P as a sum of Minkowski indecomposable polytopes.

7. Show that $\mathbb{SC}_n$ has $\binom{n}{2} 2^{n-2}$ facets. Give bounds on its number of rays.

Exercises

8. Show that graphical zonotopes are generalized permutahedra. Show that nestohedra are generalized permutahedra. Show that matroid polytopes are generalized permutahedra. Which are indecomposable?

9. If G is triangle-free, then $\mathbb{DC}(\mathrm{Z}_G)$ is simplicial (see lecture). Each face of $\mathbb{DC}(\mathrm{Z}_G)$ is associated to a polytope: which polytope?

10. The *weighted graphical zonotope* of a graph $G = (V, E)$ and weight (on edges) $\omega : E \to \mathbb{R}_+^*$ is $\mathrm{Z}_{G,\omega} := \sum_{(i,j)\in E} \omega(i,j)[\boldsymbol{e}_i, \boldsymbol{e}_j]$. Show that: $\{\text{zonotopes}\} \cap \{\text{generalized permutahedra}\} = \{\text{weighted graphical zonotopes}\}$. Show that the cone of weighted graphical zonotopes is a section (i.e. intersection with a linear sub-space) of the submodular cone, such that the rays of this section are rays of $\mathbb{SC}_n$. What is the dimension of this section?

11. Prove the cubic relations hold. Show that cubic relations generates all linear relations between submodular vectors.